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Adjoint symmetries for time-dependent second-order equations

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Abstract. We extend part of our previous work on autonomous second-order systems to time-dependent differential equations. The main subject of the paper concerns the notion of adjoint symmetries: they are introduced as a particular type of 1-form, whose leading coefficients satisfy the adjoint equations of the equations determining symmetry vector fields. It is shown that all interesting properties of adjoint symmetries, known from the autonomous theory, have their counterparts in the present framework. Of particular interest is a result establishing that Lagrangian systems seem to be the only ones for which there is a natural duality between symmetries and adjoint symmetries. A number of examples illustrate how the construction of adjoint symmetries of a given system can be explored in a systematic way.

1. Introduction

Some time ago, a paper by Gordon, containing a rather unfamiliar account of Noether's theorem for Lagrangian systems (Gordon 1986), inspired us to explore various ways in which a generalisation of this theorem can be conceived for non-Lagrangian systems. Noether's theorem essentially provides a mechanism for generating first integrals and, in the traditional picture, the mechanism is triggered by symmetry vector fields of the given equations. A generalisation which respects this point of view led us to the introduction of the notion of pseudo-symmetries: they constitute a class of vector fields which, under appropriate circumstances, can be related to first integrals. In Gordon's account, on the other hand, a predominant role is played by the adjoint equations of the linear variational equations of the given system. The generation of conservation laws through solutions of the adjoint equations seem to have penetrated the literature more in the context of partial differential equations (see, e.g., Gordon 1984, Olver 1986, Kosmann-Schwarzbach 1985 and Vinogradov 1984). For the particular case of second-order ordinary differential equations in normal form, represented by a vector field on the tangent bundle of a manifold, there is a very neat geometrical description: one can introduce adjoint symmetries as being a particular type of 1-form. We were able to explain the correspondence between pseudo-symmetries and such adjoint symmetries, thus linking the two different paths for generalising Noether's theorem. All of this, together with other aspects of interest, was discussed in the context of autonomous systems in Sarlet *et al* (1987).

It is now our feeling that adjoint symmetries constitute a more fundamental concept than pseudo-symmetries. One of the reasons is that pseudo-symmetries of a second-order equation field Γ are defined with respect to some 1-form belonging to the set $\mathfrak{X}_\Gamma^*$ (introduced in Sarlet *et al* 1984), whereas adjoint symmetries do not require any prerequisites. The main aim of the present paper is to discuss the related theory for time-dependent systems. In accordance with the previous remark, we want to bring adjoint symmetries to the forefront in this discussion. We will show that all interesting properties of adjoint symmetries of autonomous systems, as established in Sarlet *et al* (1987), have their counterparts in the present context. A number of examples will be worked out in the final section. They will illustrate that the search for adjoint symmetries of second-order systems can be organised in a rather systematic way and is a possible field for the development of suitable computer algebra routines. The next section contains the extension to $\mathbb{R}\times TM$ (needed later on) of various concepts and properties which for the autonomous case (i.e. on TM) were introduced in Sarlet *et al* (1984).

2. Geometrical features of time-dependent second-order systems

The second-order systems of ordinary differential equations under investigation in this paper are those which can be written in normal form, say

$$\ddot{q}^i = \Lambda^i(t, q, v) \qquad i = 1, \ldots, n.$$

Solutions of these equations correspond to integral curves (parametrised by t) of the vector field

$$\Gamma = \partial/\partial t + v^i\, \partial/\partial q^i + \Lambda^i(t, q, v)\, \partial/\partial v^i$$

on $\mathbb{R}\times TM$. We know from Crampin *et al* (1984) that the type (1, 1) tensor field which is to be regarded as the vertical endomorphism on $\mathbb{R}\times TM$ is given by

$$S = \frac{\partial}{\partial v^i} \otimes (\mathrm{d}q^i - v^i\, \mathrm{d}t).$$

Clearly, second-order equation fields are characterised by the property $S(\Gamma) = 0$. As in previous publications, we make no notational distinction between the linear action of a type (1, 1) tensor field on the module of vector fields and its dual action on the module of 1-forms. The only instance in which this is a little dangerous is when it comes to taking the composition of such tensor fields. Thus, for the action on 1-forms, we have the properties

$$S \circ \mathcal{L}_\Gamma S = S \qquad \mathcal{L}_\Gamma S \circ S = -S$$

whereas for the action on vector fields one has to reverse the signs on the right-hand sides of these relations. For a discussion of the eigenvalues and eigenspaces of the important tensor field $\mathcal{L}_\Gamma S$, we refer to Crampin *et al* (1984). For our present purposes, it suffices to remember that $\mathcal{L}_\Gamma S(\Gamma) = 0$ and

$$(\mathcal{L}_\Gamma S)^2 = \mathbf{1} - \Gamma \otimes \mathrm{d}t.$$

The generalisation to $\mathbb{R}\times TM$ of the sets $\mathfrak{X}_\Gamma$ and $\mathfrak{X}_\Gamma^*$ (as introduced in Sarlet *et al* (1984)) is a fairly straightforward matter. It was also done recently by Cariñena and Martínez (1989). Our definitions will, however, slightly differ from theirs, so this calls for a word of explanation. When looking at interesting vector fields and 1-forms, associated with a given second-order equation field Γ in the time-dependent case, there always appears to be some arbitrariness in the time component. Cariñena and Martínez selected the option of fixing this time component in their definition of $\mathfrak{X}_\Gamma$ and $\mathfrak{X}^*_\Gamma$. We prefer to keep the arbitrariness in the definition and talk about appropriate equivalence classes. This way, we stay closer to traditions of the past: many authors have, for example, in studying symmetries of Γ, dealt with equivalence classes of so-called dynamical symmetries.

Definitions. For each second-order equation field Γ, we set:

$$\mathfrak{X}_\Gamma = \{X \in \mathfrak{X}(\mathbb{R}\times TM) | S([\Gamma, X]) = 0\}$$

$$\mathfrak{X}^*_\Gamma = \{\phi \in \mathfrak{X}^*(\mathbb{R}\times TM) | \mathscr{L}_\Gamma(S(\phi)) = \phi - \langle \Gamma, \phi\rangle \, \mathrm{d}t\}.$$

These sets are not submodules of vector fields (respectively 1-forms) over the ring of C^∞ functions in the ordinary sense. As in the autonomous case (compare with Sarlet 1987), they acquire their own module structure for a $*$ product with functions f which is defined as follows:

$$\forall X \in \mathfrak{X}_\Gamma : f * X = fX + \Gamma(f)S(X)$$

$$\forall \phi \in \mathfrak{X}^*_\Gamma : f * \phi = f\phi + \Gamma(f)S(\phi).$$

Moreover, we can introduce projection operators:

$$\pi_\Gamma : \mathfrak{X}(\mathbb{R}\times TM) \to \mathfrak{X}_\Gamma \qquad X \mapsto \pi_\Gamma(X) = X + S([\Gamma, X])$$

$$\pi_\Gamma : \mathfrak{X}^*(\mathbb{R}\times TM) \to \mathfrak{X}^*_\Gamma \qquad \phi \mapsto \pi_\Gamma(\phi) = \mathscr{L}_\Gamma(S(\phi)) + \langle \Gamma, \phi\rangle \, \mathrm{d}t$$

which carry over the appropriate module structures, i.e. we have, for example,

$$\pi_\Gamma(f\phi) = f * \pi_\Gamma(\phi)$$

and similarly for vector fields.

Locally, a vector field $X \in \mathfrak{X}_\Gamma$, when expressed in terms of the basis $\{\Gamma, \partial/\partial q^i, \partial/\partial v^i\}$, takes the form

$$X = \tau\Gamma + \mu^i \frac{\partial}{\partial q^i} + \Gamma(\mu^i) \frac{\partial}{\partial v^i}$$

whereas a 1-form $\alpha \in \mathfrak{X}^*_\Gamma$ can, with respect to the dual basis $\{\mathrm{d}t, \theta^i = \mathrm{d}q^i - v^i \, \mathrm{d}t, \phi^i = \mathrm{d}v^i - \Lambda^i \, \mathrm{d}t\}$, be written as

$$\alpha = \alpha_i \phi^i + \Gamma(\alpha_i)\theta^i + \tau \, \mathrm{d}t.$$

It is the function τ in both expressions which could be normalised to zero. Our alternative is to treat $X_1, X_2 \in \mathfrak{X}_\Gamma$ as being equivalent if

$$X_1 - X_2 = f\Gamma$$

for some function f, and to say that $\alpha_1 \in \mathfrak{X}^*_\Gamma$ is equivalent to $\alpha_2 \in \mathfrak{X}^*_\Gamma$ if

$$\alpha_1 \wedge \mathrm{d}t = \alpha_2 \wedge \mathrm{d}t.$$

Note that such an equivalence is preserved under the $*$ multiplication with functions. We will, however, occasionally be led to also impose the same equivalence relation on general vector fields or 1-forms on $\mathbb{R}\times TM$ and will then denote the resulting quotient modules as $\tilde{\mathfrak{X}}(\mathbb{R}\times TM)$ and $\tilde{\mathfrak{X}}^*(\mathbb{R}\times TM)$.

Observe from the coordinate expression that the set $\mathfrak{X}_\Gamma$ contains the prolongations of vector fields on $\mathbb{R}\times M$ and also all dynamical symmetries of Γ, i.e. the vector fields X satisfying $[X,\Gamma]=h\Gamma$ for some function h. It is interesting, however, to look at the geometrical interpretation of $\mathfrak{X}_\Gamma$ in some more detail. From the expression of S it is clear that the statement $X\in\mathfrak{X}_\Gamma$ is equivalent to saying that $\langle\mathscr{L}_X\Gamma,\theta^i\rangle=0$. This in turn means that the flow of X maps integral curves of Γ into curves lifted from the base manifold $\mathbb{R}\times M$. All integral curves of Γ are lifted curves and a dynamical symmetry X maps them into themselves. Therefore, a dynamical symmetry X certainly belongs to $\mathfrak{X}_\Gamma$ and since $\langle\mathscr{L}_{[X,\Gamma]}\Gamma,\theta^i\rangle$ is trivially zero, the same is true for $\mathscr{L}_X\Gamma$. Conversely, suppose that $X\in\mathfrak{X}_\Gamma$ and $\mathscr{L}_\Gamma X\in\mathfrak{X}_\Gamma$, i.e. we have

$$\langle\mathscr{L}_X\Gamma,\theta^i\rangle=0 \qquad \langle\mathscr{L}_{[\Gamma,X]}\Gamma,\theta^i\rangle=0.$$

The second of these is equivalent to

$$\langle\mathscr{L}_\Gamma(\mathscr{L}_X\Gamma),\theta^i\rangle=0$$

which, in view of the first property, implies

$$\langle\mathscr{L}_X\Gamma,\mathscr{L}_\Gamma\theta^i\rangle=\langle\mathscr{L}_X\Gamma,\phi^i\rangle=0.$$

The conclusion is that $\mathscr{L}_X\Gamma=h\Gamma$ for some function h, i.e. X is a dynamical symmetry of Γ. We thus have proved the following result.

Proposition 1. A vector field X on $\mathbb{R}\times TM$ is a dynamical symmetry of Γ if and only if we have $X\in\mathfrak{X}_\Gamma$ and $\mathscr{L}_\Gamma X\in\mathfrak{X}_\Gamma$.

The zero class in the quotient structure induced by our equivalence relation on vector fields consists of the multiples of Γ (note in passing that $f\Gamma=f*\Gamma$): they are what one calls the trivial dynamical symmetries of Γ. The zero class for the quotient structure on $\mathfrak{X}^*_\Gamma$ consists of multiples of $\mathrm{d}t$. We will see in the next section that these are in fact the trivial adjoint symmetries of Γ. For the time being, the relevance of the set $\mathfrak{X}^*_\Gamma$ can be best illustrated by showing how the special case of a regular Lagrangian system fits nicely into this framework. To this end, we note that the defining relation of $\mathfrak{X}^*_\Gamma$ can equivalently be rewritten in the form

$$i_\Gamma(\mathrm{d}S(\phi)+\phi\wedge\mathrm{d}t)=0$$

which reminds us of the way time-dependent Lagrangian systems are defined via the kernel of a Cartan 2-form. In fact, for $\phi=\mathrm{d}L$, the 2-form in the above expression is precisely the Cartan 2-form $\mathrm{d}\theta_L$, so regular Lagrangian systems are those second-order equation fields Γ for which the set $\mathfrak{X}^*_\Gamma$ contains an exact 1-form ϕ with the property that $\mathrm{d}S(\phi)+\phi\wedge\mathrm{d}t$ is a contact form, i.e. a closed 2-form of maximal rank. The local existence of a Lagrangian is guaranteed if we content ourselves with a closed 1-form in $\mathfrak{X}^*_\Gamma$. It is important to realise, however, that more general systems of physical interest easily find their place in our construction. Specifically, Lagrange's equations for systems with non-conservative forces are characterised by the appearance of an element in $\mathfrak{X}^*_\Gamma$, which is the sum of an exact form and a semi-basic one. In coordinates, we can, for example, write such a form, remembering that there is some arbitrariness in the $\mathrm{d}t$ part, as follows:

$$\phi=\mathrm{d}L+Q_i(t,q,v)(\mathrm{d}q^i-v^i\,\mathrm{d}t)$$

where the functions Q_i represent the non-conservative forces. We then have

$$\mathscr{L}_\Gamma(S(\phi))-\phi=\left[\Gamma\left(\frac{\partial L}{\partial v^i}\right)-\frac{\partial L}{\partial q^i}-Q_i\right](\mathrm{d}q^i-v^i\,\mathrm{d}t)-\Gamma(L)\,\mathrm{d}t$$

from which it is easily seen that the requirement $\phi\in\mathscr{X}^*_\Gamma$ effectively means that

$$\Gamma\left(\frac{\partial L}{\partial v^i}\right)-\frac{\partial L}{\partial q^i}=Q_i.$$

In these more general situations, the 2-form $\mathrm{d}S(\phi)+\phi\wedge\mathrm{d}t$ could somehow be regarded as a pre-Cartan form. It is therefore of interest to introduce the following notion of regularity.

Definition. A 1-form $\phi\in\mathscr{X}^*_\Gamma$ is said to be non-degenerate if $\mathrm{d}S(\phi)+\phi\wedge\mathrm{d}t$ has maximal rank.

For a non-degenerate ϕ, $\mathrm{d}S(\phi)+\phi\wedge\mathrm{d}t$ has a one-dimensional kernel which therefore consists precisely of the multiples of Γ.

3. Adjoint symmetries, first integrals and Lagrangians

For later use, we first look in some more detail at the question of the existence (locally) of a Lagrangian for a given second-order equation field Γ. The following lemma treats this inverse problem question in a slightly more general way than was done by Cariñena and Martínez (1989).

Lemma. Let $\phi\in\mathscr{X}^*_\Gamma$ be non-degenerate and satisfy the condition $\mathrm{d}\phi\wedge\mathrm{d}t=0$. Then Γ represents a locally Lagrangian system. More specifically, locally there exists an exact 1-form ϕ' in $\mathscr{X}^*_\Gamma$, which is equivalent to ϕ.

Proof. Putting $\Omega=\mathrm{d}S(\phi)+\phi\wedge\mathrm{d}t$, it is easy to see (for example in coordinates) that Ω vanishes on any pair of vector fields which are vertical over $\mathbb{R}\times M$. By the fact that ϕ belongs to $\mathscr{X}^*_\Gamma$ we have $i_\Gamma\Omega=0$ and the extra assumption on ϕ means that $\mathrm{d}\Omega=0$. Hence, the Helmholtz conditions for the existence of a Lagrangian, as described, for example, in Crampin *et al* (1984), are satisfied. For the explicit appearance of the Lagrangian L, we first write ϕ as

$$\phi=\phi^{(1)}+\lambda\,\mathrm{d}t\qquad\text{with }\langle\partial/\partial t,\phi^{(1)}\rangle=0.$$

It is then clear that the assumption $\mathrm{d}\phi\wedge\mathrm{d}t=0$ implies $\mathrm{d}_x\phi^{(1)}=0$, where d_x is a formal notation for the exterior derivative on TM with the variable t regarded as a parameter. It follows that locally $\phi^{(1)}=\mathrm{d}_xL$ for some function $L(t,q,v)$. Putting

$$\phi'=\phi+\left(\frac{\partial L}{\partial t}-\lambda\right)\mathrm{d}t$$

we obtain a non-degenerate element of $\mathscr{X}^*_\Gamma$ which is equivalent to the original ϕ and is exact: $\phi'=\mathrm{d}L$. □

We now come to the introduction of the notion of adjoint symmetries. The definition below is inspired by the previous work on autonomous systems and by the result of proposition 1.

Definition. An adjoint symmetry of a second-order equation field Γ on $\mathbb{R}\times TM$ is a 1-form $\alpha \in \mathfrak{X}_\Gamma^*$, whose Lie derivative with respect to Γ again belongs to $\mathfrak{X}_\Gamma^*$.

To justify the terminology, let us look at the adjoint symmetry requirements in coordinates. If we write $\alpha \in \mathfrak{X}_\Gamma^*$ as before: $\alpha = \alpha_i \phi^i + \Gamma(\alpha_i)\theta^i + \tau\, \mathrm{d}t$, we have

$$\mathcal{L}_\Gamma \alpha = \left(2\Gamma(\alpha_i) + \alpha_j \frac{\partial \Lambda^j}{\partial v^i}\right)\phi^i + \left(\Gamma\Gamma(\alpha_i) + \alpha_j \frac{\partial \Lambda^j}{\partial q^i}\right)\theta^i + \Gamma(\tau)\, \mathrm{d}t.$$

From this we see that $\mathcal{L}_\Gamma \alpha \in \mathfrak{X}_\Gamma^*$ if and only if

$$\Gamma\Gamma(\alpha_i) + \Gamma\left(\alpha_j \frac{\partial \Lambda^j}{\partial v^i}\right) - \alpha_j \frac{\partial \Lambda^j}{\partial q^i} = 0 \qquad i = 1, \ldots, n.$$

This is a system of second-order partial differential equations for the functions α_i. Along solutions of the given differential equations (associated with Γ), they reduce to the adjoint equations (in the ordinary sense) of the linear variational equations of the given system.

In the autonomous case, there was a bijective correspondence between adjoint symmetries and invariant 1-forms, defined by the tensor $\mathcal{L}_\Gamma S$. The situation in the time-dependent framework is quite similar, except that the invariant 1-forms in question enjoy a further property: their contraction with Γ gives zero. There are various names around for the important class of 1-forms of this type. In the context of the geometry of foliations, they would be called basic forms, where basic then refers to the quotient structure induced by the integral curves of Γ. To avoid confusion with the notion of semi-basic forms on a tangent bundle, we introduce the following terminology.

Definition. A 1-form β on $\mathbb{R}\times TM$ is said to be Γ-basic if $i_\Gamma \beta = 0$ and $i_\Gamma\, \mathrm{d}\beta = 0$.

Equivalently, β is Γ-basic if $i_\Gamma \beta = 0$ and $\mathcal{L}_\Gamma \beta = 0$, from which it is clear that a Γ-basic form is the pull back of a form on the quotient space $\mathbb{R}\times TM/\Gamma$. The same kind of forms are called absolute integral invariants in the work of Cartan (see, e.g., Godbillon, 1969).

Proposition 2. The tensor field $\mathcal{L}_\Gamma S$ determines a bijection between the set of equivalence classes of adjoint symmetries and the set of Γ-basic forms.

Proof. Assume first that α is an adjoint symmetry. We thus know that $\mathcal{L}_\Gamma \alpha$ belongs to $\mathfrak{X}_\Gamma^*$, which by definition means that

$$\mathcal{L}_\Gamma(S(\mathcal{L}_\Gamma \alpha)) = \mathcal{L}_\Gamma \alpha - \langle \Gamma, \mathcal{L}_\Gamma \alpha\rangle\, \mathrm{d}t$$

or

$$\mathcal{L}_\Gamma(\mathcal{L}_\Gamma(S(\alpha)) - \mathcal{L}_\Gamma S(\alpha)) = \mathcal{L}_\Gamma \alpha - \langle \Gamma, \mathcal{L}_\Gamma \alpha\rangle\, \mathrm{d}t.$$

This can be rewritten as

$$\mathcal{L}_\Gamma(\mathcal{L}_\Gamma S(\alpha)) = \mathcal{L}_\Gamma(\mathcal{L}_\Gamma(S(\alpha)) - \alpha + \langle \Gamma, \alpha\rangle\, \mathrm{d}t).$$

Putting $\beta = \mathcal{L}_\Gamma S(\alpha)$ and remembering that $\mathcal{L}_\Gamma S(\Gamma) = 0$, we clearly have $i_\Gamma \beta = 0$, while the preceding formula, by the fact that $\alpha \in \mathfrak{X}_\Gamma^*$, further implies $\mathcal{L}_\Gamma \beta = 0$. If α_1 and α_2 are adjoint symmetries giving rise to the same β, we have $\mathcal{L}_\Gamma S(\alpha_1 - \alpha_2) = 0$, which means that α_1 and α_2 belong to the same equivalence class of forms. For the surjectivity, let β be a Γ-basic form and put $\alpha=\mathscr{L}_\Gamma S(\beta)$. We then have $\langle\Gamma,\alpha\rangle=0$ and

$$\mathscr{L}_\Gamma(S(\alpha))=\mathscr{L}_\Gamma(S(\beta))=\mathscr{L}_\Gamma S(\beta)=\alpha$$

which expresses that $\alpha\in\mathscr{X}^*_\Gamma$. Furthermore,

$$\beta=\mathscr{L}_\Gamma(S(\alpha))-S(\mathscr{L}_\Gamma\alpha)=\alpha-S(\mathscr{L}_\Gamma\alpha).$$

The invariance of β thus further implies $\mathscr{L}_\Gamma(S(\mathscr{L}_\Gamma\alpha))=\mathscr{L}_\Gamma\alpha$, which together with $\langle\Gamma,\mathscr{L}_\Gamma\alpha\rangle=0$ expresses the fact that also $\mathscr{L}_\Gamma\alpha\in\mathscr{X}^*_\Gamma$, i.e. α is an adjoint symmetry of Γ. □

As announced in the previous section, multiples of $\mathrm{d}t$ trivially satisfy the requirements for an adjoint symmetry. That they are indeed trivial is enhanced by the fact that the associated Γ-basic form is zero.

An interesting property of Γ-basic forms, which emerges from proposition 2 and has perhaps not been observed before is the following. Note for a start that $\mathscr{L}_\Gamma S$ acts as the identity on the ϕ^i components of a 1-form. Therefore, if we use β_i to denote the ϕ^i components of a Γ-basic form β, it follows from proposition 2 that they are solutions of the adjoint variational equations mentioned above.

The next results describe particular classes of adjoint symmetries which lead to the identification of a first integral or a Lagrangian for the system. Their relevance lies in the fact that they apply to any second-order equation field Γ and are in essence not more complicated than similar statements on dynamical symmetries which have been known for a long time, but only apply to systems for which a Lagrangian is known *a priori.*

Proposition 3. Let α be an adjoint symmetry of Γ such that $\mathscr{L}_\Gamma S(\alpha)=\mathrm{d}F$ for some function F. Then F is a first integral of Γ. Conversely, to every first integral there corresponds an equivalence class of adjoint symmetries determined by the same relation.

Proof. If β is a Γ-basic form which is exact, $\beta=\mathrm{d}F$ say, then obviously F is a first integral of Γ. Conversely, if $\Gamma(F)=0$, $\beta=\mathrm{d}F$ defines a Γ-basic form. The proof now is a direct consequence of proposition 2. □

Note that the simplest representative of the class of adjoint symmetries corresponding to a given first integral F (namely the one with zero $\mathrm{d}t$ part) can also be written as $\alpha=\pi_\Gamma(\mathrm{d}F)$. It is this property which also remains of particular interest when F is not a first integral.

Proposition 4. Let α be an adjoint symmetry of Γ such that $\alpha=\pi_\Gamma(\mathrm{d}F)$ for some function F. Then, $\Gamma(F)$ is a Lagrangian for the given system. Conversely, if $\Gamma(F)$ is a Lagrangian, $\pi_\Gamma(\mathrm{d}F)$ is an adjoint symmetry of Γ.

Proof. The extra assumption on α means that

$$\alpha=\mathscr{L}_\Gamma S(\mathrm{d}F)+S(\mathrm{d}\Gamma(F))+\langle\Gamma,\mathrm{d}F\rangle\,\mathrm{d}t.$$

Applying $\mathscr{L}_\Gamma S$ to both sides, we find

$$\mathscr{L}_\Gamma S(\alpha)=\mathrm{d}F-\langle\Gamma,\mathrm{d}F\rangle\,\mathrm{d}t-S(\mathrm{d}\Gamma(F)).$$

Since α is an adjoint symmetry, this must be an invariant 1-form. Thus we have

$$0=\mathrm{d}\Gamma(F)-\langle\Gamma,\mathrm{d}\Gamma(F)\rangle\,\mathrm{d}t-\mathscr{L}_\Gamma(S(\mathrm{d}\Gamma(F)))$$

which by definition means that $\mathrm{d}\Gamma(F)$ belongs to $\mathfrak{X}^*_\Gamma$, and therefore $\Gamma(F)$ is a Lagrangian. The converse statement is a matter of reading these formulae in reverse order. □

As in the autonomous case, we now come to a notion of self-adjointness of a second-order equation field Γ, which will turn out to be relevant for understanding how the above results relate to the previously known properties of dynamical symmetries of Lagrangian systems. Let ϕ be a non-degenerate element of $\mathfrak{X}^*_\Gamma$, so that the kernel of the 2-form $\Omega=\mathrm{d}S(\phi)+\phi\wedge\mathrm{d}t$ is the one-dimensional distribution spanned by Γ. By contraction, we obtain a map from vector fields to 1-forms which takes the same value on all vector fields of the same equivalence class. A 1-form β is in the image of this map if and only if $\langle\Gamma,\beta\rangle=0$. Forms β with this property are characterised by the fact that they can be written as $\beta=\mathscr{L}_\Gamma S(\alpha)$ for some α and two α giving birth to the same β are equivalent in the sense of section 2. We are thus led to considering rather the classes $[X]$ and $[\alpha]$ as prime objects. In other words, we focus on the linear map $\gamma\colon\tilde{\mathfrak{X}}(\mathbb{R}\times TM)\to\tilde{\mathfrak{X}}^*(\mathbb{R}\times TM)$, defined by

$$\gamma:[X]\mapsto[\alpha]\qquad\text{with }\alpha=\mathscr{L}_\Gamma S(i_X\Omega).$$

It is easy to verify that γ is indeed an isomorphism between both quotient modules, the inverse map being defined by $\gamma^{-1}:[\alpha]\mapsto[X]$, with X any vector field satisfying $i_X\Omega=\mathscr{L}_\Gamma S(\alpha)$.

Definition. A second-order equation field Γ is said to be self-adjoint if there exists a non-degenerate $\phi\in\mathfrak{X}^*_\Gamma$, such that the associated map γ is a bijection between equivalence classes of dynamical symmetries and equivalence classes of adjoint symmetries.

Proposition 5. Γ is self-adjoint if and only if Γ is locally Lagrangian.

Proof. Consider the relation (for some $\phi\in\mathfrak{X}^*_\Gamma$):

$$i_X(\mathrm{d}S(\phi)+\phi\wedge\mathrm{d}t)=\beta.$$

We then have

$$\begin{aligned}\mathscr{L}_\Gamma\beta&=i_X(\mathrm{d}\mathscr{L}_\Gamma(S(\phi))+\mathscr{L}_\Gamma\phi\wedge\mathrm{d}t)+i_{[\Gamma,X]}(\mathrm{d}S(\phi)+\phi\wedge\mathrm{d}t)\\&=i_X(\mathrm{d}\phi+(i_\Gamma\,\mathrm{d}\phi)\wedge\mathrm{d}t)+i_{[\Gamma,X]}(\mathrm{d}S(\phi)+\phi\wedge\mathrm{d}t).\end{aligned}$$

If X is a dynamical symmetry of Γ, the second term in this expression vanishes identically. By definition, Γ is self-adjoint if for some appropriate ϕ we have $\mathscr{L}_\Gamma\beta=0$ for all dynamical symmetries. It follows that Γ is self-adjoint if and only if

$$i_X(\mathrm{d}\phi+(i_\Gamma\,\mathrm{d}\phi)\wedge\mathrm{d}t)=0$$

for all dynamical symmetries X. Now, in a coordinate chart which is chosen to straighten out Γ, all coordinate vector fields are dynamical symmetries. Therefore, we must actually have

$$\mathrm{d}\phi+(i_\Gamma\,\mathrm{d}\phi)\wedge\mathrm{d}t=0$$

which is equivalent to $\mathrm{d}\phi\wedge\mathrm{d}t=0$. By the lemma at the beginning of this section we conclude that ϕ can be replaced (locally) by an equivalent $\phi'\in\mathfrak{X}^*_\Gamma$ which is exact and thus that Γ is locally Lagrangian. □

Some comments are in order now. For a general second-order equation field Γ, interesting statements about adjoint symmetries cannot be translated to dual statements for dynamical symmetries. Proposition 5 indicates that such a translation only works properly for Lagrangian systems. If, then, the system is known to be Lagrangian from the outset, it is clear that the translation of proposition 3 is just Noether's theorem. It is not difficult to verify by some coordinate calculations that the translation of proposition 4 under the same circumstances yields a result derived by Prince concerning the existence of an alternative Lagrangian (Prince 1983). At the same time, since our current results do not require the knowledge of a Lagrangian, it is clear that we do have a proper generalisation here.

4. Recovering the autonomous case

The geometry of the odd-dimensional manifold $\mathbb{R}\times TM$ is in many respects different from the geometry of TM. From a purely analytical point of view, however, an autonomous system of second-order differential equations is, on the surface, an obvious special case of a non-autonomous one. It is therefore of some interest to have a look at the way the present formalism reduces to the one on TM when we are dealing with autonomous differential equations. For that purpose, it suffices to see what happens with the sets $\mathfrak{X}^*_\Gamma$ and $\mathfrak{X}_\Gamma$.

It is clear that we can write

$$S=S_0-\Delta\otimes \mathrm{d}t \qquad \text{with } S_0=\partial/\partial v^i\otimes \mathrm{d}q^i \quad \Delta=v^i\,\partial/\partial v^i.$$

For an autonomous system it further makes sense to decompose Γ as

$$\Gamma=\partial/\partial t+\Gamma_0.$$

Let us then start by looking, within the present formalism, at elements ϕ of $\mathfrak{X}^*_\Gamma$ which satisfy $\mathscr{L}_{\partial/\partial t}\phi=0$. We then have

$$\mathscr{L}_\Gamma(S(\phi))=\mathscr{L}_{\Gamma_0}(S_0(\phi))-\mathscr{L}_{\Gamma_0}(\langle\Delta,\phi\rangle)\,\mathrm{d}t$$

and it readily follows that the defining relation of $\mathfrak{X}^*_\Gamma$ splits into two parts. The first one, obtained by taking the contraction with $\partial/\partial t$, is

$$i_{\Gamma_0}(\phi-\mathrm{d}\langle\Delta,\phi\rangle)=0$$

and subtracting this from the full relation leads to

$$\mathscr{L}_{\Gamma_0}(S_0(\phi))=\phi.$$

At this stage ϕ is, in principle, still allowed to have a $\mathrm{d}t$ term, but one easily sees that such a term does not contribute to any of these conditions and may thus safely be omitted. This being done, the second condition is precisely the defining relation of $\mathfrak{X}^*_{\Gamma_0}$ on TM, while the first one is a consequence of it (see proposition 5.1 in Sarlet *et al* (1984)).

The situation is slightly different for the vector fields. Suppose that we similarly start looking at elements X of $\mathfrak{X}_\Gamma$ which satisfy $\mathscr{L}_{\partial/\partial t}X=0$. The defining relation of $\mathfrak{X}_\Gamma$ then becomes

$$0=S([\Gamma,X])=S_0([\Gamma_0,X])-\langle[\Gamma_0,X],\mathrm{d}t\rangle\Delta.$$

If we want X to be an object which properly lives on TM, the second term on the right-hand side will disappear and we are back to the definition of $\mathfrak{X}_{\Gamma_0}$ on TM. In principle, however, we could allow for vector fields X with a non-zero $\partial/\partial t$ term and seemingly have a wider range of possibilities at our disposal. This should not come as a surprise, because this kind of situation is well known in the study of symmetries. As a matter of fact, even if the given system is autonomous, one can still envisage symmetry transformations which change the parametrisation, i.e. look at dynamical symmetries rather than strict symmetries. This is in effect what is usually done in calculating all point symmetries of a given second-order system. However, we also know about the equivalence relation by which every local result concerning a vector field with non-vanishing $\partial/\partial t$ term can be translated to a corresponding result for a vector field without such a term. This means that as long as one restricts all coefficients of X (or in fact any other object of interest) to be time-independent, nothing is lost by working in the proper TM configuration. Needless to say, it can also be of interest, even for autonomous systems, to work in the $\mathbb{R}\times TM$ configuration, because for example, autonomous differential equations may well have time-dependent first integrals.

5. Illustrative examples

Let us start with some comments concerning a well established subject: the search for symmetries of differential equations and its automatisation via a suitable computer algebra package. Limiting ourselves to the case of second-order ODE of the form $\ddot{q}^i = \Lambda^i(t, q, \dot{q})$, a large part of the existing literature on symmetries is restricted to what we call point symmetries, i.e. dynamical symmetries of Γ which are prolongations of a vector field $X^{(0)} = \tau(t, q)\partial/\partial t + \xi^i(t, q)\partial/\partial q^i$ on $\mathbb{R}\times M$. The search for solutions will, most of the time, be conducted along the lines of Lie's original theory and will accordingly involve the computation of the second prolongation of $X^{(0)}$ (see, e.g., Olver (1986) as a general reference). As far as we know, the existing computer algebra implementations of the search for symmetries follow the same pattern and are always restricted to the case of point symmetries (this is afterall not surprising as these packages are equally applicable to the much wider class of PDE).

Within the present context of second-order ODE an alternative formulation of the problem of constructing symmetries is simply to search for solutions μ^i of the system of equations

$$\Gamma^2(\mu^i) - \frac{\partial\Lambda^i}{\partial v^j}\Gamma(\mu^j) - \frac{\partial\Lambda^i}{\partial q^j}\mu^j = 0 \qquad i = 1, \ldots, n.$$

The case of point symmetries corresponds to solutions of the restricted form

$$\mu^i = \xi^i(t, q) - v^i\tau(t, q).$$

In most applications, the given functions Λ^i will have a polynomial dependence on the velocity variables v^i. It should therefore be just as straightforward (though possibly much more tedious) to search for dynamical symmetries starting from a more general ansatz for the functions μ^i, namely a polynomial in the v^j of arbitrary preassigned degree. In addition, it must be fairly simple to extend existing computer algebra packages for that purpose.

Now for the present paper, where the study of adjoint symmetries is the central theme, the program for applications is of course quite similar. We are interested in solutions of the system of equations

$$\Gamma^2(\alpha_i)+\Gamma\left(\frac{\partial\Lambda^j}{\partial v^i}\alpha_j\right)-\frac{\partial\Lambda^j}{\partial q^i}\alpha_j=0 \qquad i=1,\ldots,n.$$

Here again, the search for particular solutions can be gradually stepped up, from solutions depending on t and q only, to polynomials in the v^j of any degree. It is of some interest to have another look here at the duality between symmetries and adjoint symmetries which exists for Lagrangian systems. It is easy to verify that the essential part of the map γ for the case of a regular Lagrangian system is determined by the relation

$$\frac{\partial^2 L}{\partial v^i\partial v^j}\mu^j=-\alpha_i.$$

In most standard applications, L will be quadratic in the velocities, so that solutions α_i of the adjoint equations will have the same polynomial structure as corresponding solutions μ^j of the symmetry conditions. This means in particular that in order to cover all cases of point symmetries in the dual picture, solutions for α_i must also be allowed to depend linearly on the v^j. When no Lagrangian for the system is known *a priori*, we have learned that interesting things happen whenever an adjoint symmetry α matches the supplementary requirement $\alpha=\pi_\Gamma(\mathrm{d}F)$. Essentially, this requires the coefficients α_i to be of the form $\alpha_i=\partial F/\partial v^i$ for some function F.

It is to be expected that interesting functions F may well depend quadratically on the velocities, so that also from this point of view there is an incentive to push the search for solutions of the adjoint equations at least to the first-degree level. Work concerning the development of computer algebra procedures along these lines is under way. Meanwhile, the examples which follow were computed by hand.

As a first example, consider the Emden equation

$$\ddot q=-\frac{2}{t}\dot q-q^5.$$

Using a to denote the coefficient of $\mathrm{d}v$ of an adjoint symmetry α, the equation to be solved becomes

$$\Gamma^2(a)+\Gamma\left(-\frac{2}{t}a\right)+5aq^4=0.$$

One readily verifies that there are no solutions depending on t and q only. For the next step in a systematic search we require a to be of the form $a(t,q,v)=a_1(t,q)v+a_0(t,q)$. The coefficients of different powers of v then give rise to the following system of equations:

$$\frac{\partial^2 a_1}{\partial q^2}=0 \qquad 2\frac{\partial^2 a_1}{\partial t\,\partial q}+\frac{\partial^2 a_0}{\partial q^2}-\frac{8}{t}\frac{\partial a_1}{\partial q}=0$$

$$\frac{\partial^2 a_1}{\partial t^2}+2\frac{\partial^2 a_0}{\partial t\,\partial q}-\frac{6}{t}\frac{\partial a_1}{\partial t}+\frac{12}{t^2}a_1-3q^5\frac{\partial a_1}{\partial q}-\frac{4}{t}\frac{\partial a_0}{\partial q}=0$$

$$\frac{\partial^2 a_0}{\partial t^2}-2q^5\frac{\partial a_1}{\partial t}-q^5\frac{\partial a_0}{\partial q}+\frac{2}{t^2}a_0-\frac{2}{t}\frac{\partial a_0}{\partial t}+\frac{4}{t}q^5a_1+5q^4a_0=0.$$

It is found to have the unique solution

$$a = 2t^3 v + t^2 q.$$

The corresponding adjoint symmetry appears to match the conditions of proposition 3 and produces in this way the well known integral

$$F = t^3(v^2 + \tfrac{1}{3}q^6) + t^2 qv.$$

Consider next another simple one-degree-of-freedom system, determined by

$$\Lambda = \frac{q}{t^2}\, v - \frac{1}{t}\, v^2.$$

This time, the adjoint symmetry condition has a solution depending on t and q only, namely

$$a(t, q) = e^{q/t}.$$

Again it gives rise to a first integral, having the form

$$F = e^{q/t} v.$$

A Lagrangian for this equation is not readily available, so it is interesting to compare the situation here with information which can be gathered from the search for symmetries. The equation under consideration is discussed in Olver (1986) as example 2.58. It has the point symmetry $t\, \partial/\partial t + q\, \partial/\partial q$. In adapted coordinates which straighten out this symmetry vector field, one is reduced to a first-order differential equation of Riccati type. It appears that it is only at this secondary stage that a symmetry of the reduced equation produces a first integral equivalent to the function F above.

At this point, it is worthwhile mentioning the following general property of adjoint symmetries, which can very easily be verified. If α is an adjoint symmetry of Γ and f is a first integral, then $f\alpha$ is another adjoint symmetry. For the present example, we conclude that $a = e^{2q/t} v$ determines an adjoint symmetry of degree 1 in v (it appears to be the only one). Obviously, this new adjoint symmetry will also match the requirements of proposition 3, but the corresponding first integral will be a function of the original one ($\frac{1}{2}F^2$ in this case). Note finally that division of the original a by F shows that $a = 1/v$ also defines an adjoint symmetry. This remark illustrates that it may be of interest in certain cases to extend the search for adjoint symmetries from polynomial to rational expressions in the velocities.

Referring to the discussion at the end of the preceding section, we now want to treat an example of an autonomous system within the time-dependent framework. Consider the following three-degree-of-freedom system:

$$\ddot{q}^1 = -q^1 \dot{q}^3 \qquad \ddot{q}^2 = -q^2 \dot{q}^3 \qquad \ddot{q}^3 = q^1 \dot{q}^1 + q^2 \dot{q}^2.$$

The computations of course become quite labourious here, so we restrict ourselves to a listing of the results obtained to first degree in the velocities. There are four adjoint symmetries $\alpha^{(i)}$, two of which have leading coefficients $\alpha_j^{(i)}$ which are independent of the velocities. As the corresponding Γ-basic forms $\beta^{(i)}$ have the same dv^j coefficients, it is not necessary to list both expressions. Three of the Γ-basic forms are given by

$$\beta^{(1)} = -dv^3 + q^1\, dq^1 + q^2\, dq^2$$

$$\beta^{(2)} = q^1\, dv^2 - q^2\, dv^1 + v^2\, dq^1 - v^1\, dq^2$$

$$\beta^{(3)} = v^1\, dv^1 + v^2\, dv^2 + v^3\, dv^3.$$

It is clear that these are exact forms; they give rise to the first integrals

$$F_1 = \tfrac{1}{2}[(q^1)^2 + (q^2)^2] - v^3$$

$$F_2 = q^1 v^2 - q^2 v^1$$

$$F_3 = \tfrac{1}{2}[(v^1)^2 + (v^2)^2 + (v^3)^2].$$

For the fourth adjoint symmetry, we have

$$\alpha_j^{(4)} = q^j + tv^j.$$

The first three $\alpha^{(i)}$ obviously would have been found also in the purely autonomous framework on TM, but $\alpha^{(4)}$ is an extra result obtained via the time-dependent setup on $\mathbb{R} \times TM$. The Γ-basic form $\beta^{(4)}$ is not exact. However, it is clear that we have $\alpha_j^{(4)} = \partial F_4 / \partial v^j$ (meaning that $\alpha^{(4)} = \pi_\Gamma(\mathrm{d}F_4)$), with

$$F_4 = \tfrac{1}{2}t[(v^1)^2 + (v^2)^2 + (v^3)^2] + q^i v^i.$$

Hence, in agreement with proposition 4, $\Gamma(F_4)$ is a Lagrangian for the given system. Dividing by 3 and subtracting a suitable total time derivative, this Lagrangian takes the standard form

$$L = \tfrac{1}{2}[(v^1)^2 + (v^2)^2 + (v^3)^2] - \tfrac{1}{2}[(q^1)^2 + (q^2)^2]v^3.$$

With the help of this Lagrangian it is again possible, if desired, to translate the above results to the context of symmetries. The adjoint symmetries $\alpha^{(i)}$ then all turn out to correspond to point symmetries of Γ.

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